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Scalar Manifolds and Jordan Pairs in Supergravity

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It is shown how the concept of a Jordan pair, which generalizes that of Jordan algebra and links Jordan algebras to Lie algebras, enters a model for supergravity studied recently by Günaydin, Sierra, and Townsend, called "magical supergravity." This model is very briefly reviewed, as are the reasons that led, starting from the theory of Jordan algebras, to the definition and development of Jordan pairs. Examples of Jordan pairs are given, which show the beauty, simplicity, and usefulness of such objects.

1. INTRODUCTION

One of the most appealing features of supersymmetry is that it leads naturally to supergravity, once realized locally (Cremmer, 1982; van Nieuwenhuizen, 1981). This is rather intuitive if one thinks of the algebra of supersymmetry generators:

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{1}$$

where $i, j = 1, \ldots, N$ are the indices for the extended theory. Since the combination of two supersymmetry transformations amounts to moving from one point to another point in spacetime, local supersymmetry implies that such translation be related to a spacetime-dependent parameter, which effectively means performing a general coordinate transformation. The graviton, being the gauge field of general coordinate transformations, therefore enters naturally in the theory, together with a dimensioned gravitational coupling constant and with its fermionic partner, the spin-$\frac{3}{2}$ gravitino, the gauge field of local supersymmetry transformations. Therefore one could start from a theory realized just in terms of one graviton and one gravitino, which is referred to as $N = 1$ "pure supergravity." There are three ways—plus combinations of these—of adding matter multiplets to pure supergravity:

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1. Add an arbitrary number of matter fields, that is, vector or chiral multiplets (Ferrara, and Fayet, 1977) with their internal symmetry quantum numbers (Ferrara *et al.*, 1976).

2. Extend the supersymmetry to $N>1$ (spinorial) supersymmetry generators ($N=8$ being the upper limit for the theory so as not to yield particles of spin higher than 2) (Cremmer and Julia, 1978, 1979); in this case indeed fields of spin lower than $\frac{3}{2}$ appear in the graviton multiplet.

3. Extend the dimensionality of spacetime (à la Kaluza–Klein) to a dimension $d>4$ ($d<11$ being the analogous limitation as in the previous case) and then proceed to the reduction of the theory to four dimensions, by a procedure called spontaneous compactification (Cremmer and Sherk, 1977); in this case the components of the graviton and gravitino related to the extra coordinates constitute, in four dimensions, fields of lower spin.

The present paper concerns the reduction to four dimensions of a theory, extensively worked out by Günaydin, Sierra, and Townsend (GST) (Günaydin *et al.* (1983, 1984), which combines the three above ways of introducing matter fields in the theory. The starting point of the work of GST is the formulation of $N=2$, $d=5$ supergravity coupled to n Maxwell (i.e., Abelian) vector fields. Two important features of this theory are the unification of gravity with an electromagnetic type of interaction and, most interestingly in my point of view, the existence of an "exceptional case" which (contrary to all other cases) cannot be derived from a suitable $N=8$ theory—that is to say, another upper limit beyond the usual $N=8$ case. The exceptional case is related to exceptional symmetry groups for the Lagrangian: one more reason for becoming interested in such models, given the growing success of theories [in particular the superstring theory (Green and Schwarz, 1984; Candelas *et al.*, 1985)] involving the exceptional groups (Jacobson, 1971; Gürsey 1975, 1977, 1978; Günaydin *et al.*, 1978). The GST theory will be briefly outlined in Section 3.

The object of investigation of the present paper is the manifold parametrized by the scalar fields of such a theory. The motivation for studying in particular the scalar manifolds in extended supergravity is given in the next section, where I introduce these manifolds in a brief and informal review.

The work by Günaydin, Sierra, and Townsend put very nicely into an algebraic form all the constraints characterizing the geometry of the scalar fields of their model. In a recent paper (Truini *et al.*, 1985) we pursued this algebraic approach by emphasizing that there is a very interesting mathematical structure (which is quite new to physicists) underlying the GST theory in four dimensions: the structure of a Jordan pair. Besides reviewing the entire subject, the present paper aims at indicating the usefulness and naturalness of implementing the Jordan pair language in such a theory. Not quite an algebra, but not far from it, the Joran pair concept nicely generalizes, with far-reaching consequences, that of a Jordan algebra, which has had so much development in both physics and mathematics. The definition and the relevant characteristics of Jordan pairs will be introduced in Section 4. In Section 5 I show how Jordan pairs enter the "magical supergravity" theory.

2. THE SCALAR MANIFOLD

A property of extended supergravity is the occurrence of noncompact global symmetries and of compact local symmetries in the Lagrangian or in the equations of motion. The physical meaning of these symmetries is not clear, but their importance lies in the fact that they strongly constrain the form of the Lagrangian and, in particular, give a geometrical interpretation to the scalar fields of the theory. The scalar fields assume, indeed, particular interest in the theories in which they are coupled to gravity, due to the fact that they can (and do) appear nonpolynomially in the Lagrangian, since the terms of the type $K\phi$ (where K is the gravitational coupling constant and ϕ is a scalar field) are dimensionless (Cremmer, 1982; van Nieuwenhuizen, 1981).

If we denote by G the noncompact group related to the global symmetry and by H the compact group related to the local symmetry, then it is assumed, in extended supergravity, that the scalars parametrize the coset space G/H. It is further assumed that H is the maximal compact subgroup of G, in order to have a positive-definite metric on the coset; this implies that the kinetic terms for the scalar fields be positive-definite ("no ghosts in the theory").

The best way to realize how such a type of symmetry occurs is to start from a theory in $d>4$ dimensions and perform the dimensional reduction. One thus realizes that the global symmetries can be related to the reparametrization of the extra coordinates—hence they are of noncompact type—whereas the local symmetries can be related to the extension of the local Lorentz symmetry to the extra dimensions—the metric in the extra dimensions being such that the resulting symmetry be of compact type. Examples of these theories, with particular emphasis on the so called "hidden symmetries" of the scalar fields, can be found in Cremmer (1982). It is interesting to notice that the more the symmetry is extended, the more restrictions are put into the theory, so far as the possible hidden symmetries—hence the allowed geometries for the scalar fields—are concerned. It is currently believed that in the maximally extended theory ($d=4$, $N=8$ or $d=11$, $N=1$), the manifold of scalars is the unique geometry of the coset space $E_{7(+7)}/SU(8)$ (Cremmer and Julia, 1979). Notice, in fact, that because of dimensional reduction, from 11 to 4 dimensions, we expect the global symmetry group G to contain $Sl(7.\mathbb{R})$—related to the reparametrization in the seven extra coordinates—while, because of $N=8$, there are reasons to expect an $SU(8)$ maximal local symmetry (Cremmer, 1982). Since the group $H=SU(8)$ must be the maximal compact subgroup of G and since the number of scalars $[=\dim(G)-\dim(H)]$ is 70, in $N=8$ supergravity, the exceptional group $E_{7(+7)}$ is singled out as the only simple Lie group satisfying all the requirements for the global symmetry.

The symmetry of the scalar fields extends (possibly only on-shell) to the other fields in the theory, putting new constraints into the various terms of the Lagrangian. From these facts—i.e. from the peculiarity of the geometry of the scalars and from the fundamental role that this geometry has in building the Lagrangian—stems the unique role of the scalar fields in supergravity, which motivates the study of the scalar manifolds and of the way of explicitly realizing the parametrization of such manifolds.

3. MAGICAL SUPERGRAVITY

In this section, I briefly outline the "magical" case of the Maxwell-Einstein supergravity theory worked out by GST [see Günaydin *et al.* (1983, 1984) for details]. The word "magical" comes from its connection with the Freudenthal-Tits magic square (FTMS) (Freudenthal, 1959; Tits, 1955), a table relating Lie algebras to Jordan algebras and Hurwitz algebras (Schafer, 1966), which I shall introduce later in this section.

The field content of $N=2$, $d=5$ supergravity coupled to n Maxwell (i.e., Abelian) vector multiplets is the following:

$$\{e_\mu^m, \psi_\mu^i, A_\mu^I, \lambda^{ai}, \phi^x\} \tag{2}$$

$$i=1,2; \quad I=0,1,\ldots,n; \quad a,x=1,\ldots,n; \quad m,\mu=0,\ldots,4$$

Because of $N=2$, the graviton multiplet indeed contains two gravitinos and one vector field, whereas each vector multiplet contains two spin-$\frac{1}{2}$ and one scalar field.

The problem of characterizing the scalar field geometry has been reduced by GST to a purely algebraic problem. The n-dimensional scalar manifold $\mathfrak{M}$ is viewed as a hypersurface of an $(n+1)$-dimensional Riemannian space $\mathscr{C}$ satisfying the equation

$$\mathcal{N}(\xi)=1 \tag{3}$$

where $\mathcal{N}$ is a homogeneous cubic polynomial in the $\mathscr{C}$-coordinates ξ.

The "magical case" occurs under the assumptions that $\mathcal{N}$ is not factorizable and $\mathfrak{M}$ is a locally symmetric space. Then it is shown by GST that this is only possible if the number of scalar fields is

$$n=3[1+\dim(\mathbb{A})]-1 \tag{4}$$

where $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathfrak{C}$ is one of the four Hurwitz algebras of the real, complex, quaternion, octonion (or Cayley) numbers [hence $\dim(\mathbb{A}) - 1, 2, 4, 8$, respectively, and $n = 5, 8, 14, 26$].

It is also shown that $\mathfrak{M}$ is a homogeneous space, which, for the various n values, is the following coset[2]:

$$\begin{array}{lll} \text{for } n=5 & (\text{i.e., } \mathbb{A}=\mathbb{R}) & \mathfrak{M}=Sl(3,\mathbb{R})/SO(3) \\ \text{for } n=8 & (\text{i.e., } \mathbb{A}=\mathbb{C}) & \mathfrak{M}=Sl(3,\mathbb{C})/SU(3) \\ \text{for } n=14 & (\text{i.e. } \mathbb{A}=\mathbb{H}) & \mathfrak{M}=SU^*(6)/USp(6) \\ \text{for } n=26 & (\text{i.e., } \mathbb{A}=\mathfrak{C}) & \mathfrak{M}=E_{6(-26)}/F_4 \end{array} \tag{5}$$

The relationship between this theory and Jordan algebras is the following. If we denote by M_3^i, $i = 1, 2, 4, 8$,[3] the Jordan algebra of the 3×3 Hermitian matrices (over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathfrak{C}$, respectively), then:

1. The points of $\mathscr{C}$ are elements of M_3^i.
2. $\mathscr{N}$ is the norm (generalized determinant) (Jacobson, 1968) of M_3^i.
3. The points of $\mathfrak{M}$ are the norm 1 elements of M_3^i.
4. G is the reduced structure group (norm-preserving group) of M_3^i
5. H is the automorphism group of M_3^i.

When the theory is reduced to four dimensions, the number of scalars increases, since we get $n+1$ scalars from the vector fields and 1 scalar from the graviton field, adding up to $2(n+1)$ [$=2\dim(M_3^i)$] scalars. The scalar manifolds in four dimensions are, for the various n values (i.e., $n = 5, 8, 14, 26$), the following coset spaces:

$$\begin{array}{ll} Sp(6,\mathbb{R})/SU(3)\times U(1), & SU(3,3)/SU(3)\times SU(3)\times U(1) \\ SO^*(12)/SU(6)\times U(1), & E_{7(-25)}/E_6\times U(1) \end{array} \tag{6}$$

This is the case I shall consider in the next section, but before focusing on it I report the case of the reduction of the GST theory to three dimensions in order to complete the link between this theory and the Freudenthal–Tits magic square. In three dimensions the scalars parameterize the following coset spaces, according to the values of $n = 5, 8, 14, 26$:

$$\begin{array}{ll} F_{4(4)}/USp(6)\times SU(2), & E_{6(2)}/SU(6)\times SU(2) \\ E_{7(-5)}/SO(12)\times SU(2), & E_{8(-24)}/E_7\times SU(2) \end{array} \tag{7}$$

[2] I note in passing that $SU^*(6) \approx Sl(3, \mathbb{H})$ and therefore $E_{6(-26)}$, which contains the $Sl(3, \mathbb{A})$ groups (for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) as subgroups, can be thought of as a generalization of the special linear groups to the octonions. Such a generalization has the sole meaning of an extrapolation since, due to the nonassociativity of the Cayley algebra, octonionic matrices cannot form a group under the linear matrix product.

[3] M_3^8 is called the exceptional Jordan algebra.

Table I. The Particular Real Form of the FTMS for $N=2$ Supergravity[a]

$\mathbb{A}$ \ J	M_3^1	M_3^2	M_3^4	M_3^8
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	F_4
$\mathbb{R}\oplus\mathbb{R}$	$Sl(3,\mathbb{R})$	$Sl(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
$\mathbb{H}_s$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
$\mathfrak{C}_s$	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

[a] $\mathbb{H}_s$ and $\mathfrak{C}_s$ are quaternionic and octonionic split (namely nondivision) algebras over the reals. M_3^i ($i=1,2,4,8$) is a real Jordan algebra. $\mathbb{A}$ is an alternative, composition, real algebra.

All the global symmetry groups of the "magical case"—namely all the groups appearing in the "numerator" of the cosets shown above—plus the local symmetry groups of the five-dimensional theory fit into the Freudenthal–Tits magic square (FTMS) (hence the name given to this theory) given in Table I. The recipe given by Tits (1955) for building the 16 Lie algebras L of this table is the following:

$$L = \mathrm{DER}(\mathbb{A})\oplus(\mathbb{A}_0\times J_0)\oplus\mathrm{DER}(J) \tag{8}$$

where $\mathrm{DER}(\mathbb{A})$ and $\mathrm{DER}(J)$ are the generators of the automorphism group of the Hurwitz (division) algebra $\mathbb{A}$ and of the Jordan algebra J; $\mathbb{A}_0$ and J_0 are the traceless elements of $\mathbb{A}$ (that is, the pure imaginary elements of $\mathbb{A}$) and of J. For $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathfrak{C}$ (real, complex, quaternion, octonion numbers, respectively) and from $J=M_3^i$, $i=1,2,4,8$, we get from (8) the Lie algebras shown in Table I.

As already said, not all the groups of the GST theory appear in Table I, but I stress here that all of them, modulo some $U(1)$ or $SU(2)$, are real forms of the complex Lie algebras shown in the "complex form" of the magic square of Table II. The construction of this table is the same Tits construction with the real Jordan algebras M_3^i replaced by the corresponding

Table II. The FTMS for complex Lie Algebras[a]

$\mathbb{A}$ \ J	J_3^1	J_3^2	J_3^4	J_3^8
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2\oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathfrak{C}$	F_4	E_6	E_7	E_8

[a] J_3^i, $i=1,2,4,8$, is the complexification of the 3×3 Hermitian matrices M_3^i over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathfrak{C}$, respectively.

complexification J_3^i.[4] This notation, namely M_3^i for the real Jordan algebras of the FTMS and J_3^i for the complex ones, will be kept throughout this paper.

4. INTRODUCING THE JORDAN PAIRS

Having described what is meant by "magical supergravity," I now turn to the Jordan pair content of the theory. In order to motivate the introduction of such a peculiar concept as that of a Jordan pair and at the same time make this section self-consistent, it is worthwhile to sketch briefly the development of the Jordan theory in mathematics following its creation by Jordan, von Neumann, and Wigner (Jordan, 1932; Jordan *et al.*, 1934). References to papers containing all details of what is outlined here unrigorously will be given throughout the section; I particularly recommend the excellent and brief review by McCrimmon (1978).

The Jordan algebraic approach to quantum mechanics was intended to formulate quantum theory just in terms of observables. The resulting algebra is the algebra of "formally real" Hermitian matrices—"formally real" meaning that the diagonal entries, hence the eigenvalues, of the matrices are real and therefore the algebra is real. The formal reality axiom $(x^2+y^2=0 \Rightarrow x=y=0)$ was dropped by mathematicians, who wanted to formulate the Jordan theory on a generic (numerical) field, thus retaining only the other two axioms that define Jordan algebras:

$$\begin{aligned} x\cdot y &= y\cdot x && \text{(commutativity)} \\ (x^2\cdot y)\cdot x &= x^2\cdot(y\cdot x) && \text{(power associativity)} \end{aligned} \tag{9}$$

The original algebras—those including the "formal reality" axiom—are referred to as real Jordan algebras. Jordan, von Neumann, and Wigner gave a complete classification of the latter. They all correspond to the algebras of Hermitian matrices on a real, a complex, or a quaternionic Hilbert space, with the single exception of M_3^8, the algebra of 3×3 Hermitian matrices over the octonions,[5] for which no underlying Hilbert space can be defined.

[4]There is another difference in the construction of Table II versus Table I: the algebras $\mathbb{A}$ must be carefully chosen in order to get the right noncompact real forms in Table I. For instance, in the second row the algebra $\mathbb{A}$ is $\mathbb{R}\oplus\mathbb{R}$ in Table I and $\mathbb{C}$ in Table II. The reason for this difference is that the generators built out of $\mathbb{A}\times J_0$ must be noncompact (Hermitian), and an imaginary unit (like $\mathbb{A}_0$ if $\mathbb{A}$ were $\mathbb{C}$) in front of J_0 (which is already Hermitian) would spoil the noncompactedness. This care need not be taken for the complex Lie algebras because of complex linearity.

[5]I should actually write "formally real Hermitian matrices" or "Hermitian matrices over the real octonions" in order to stress that the diagonal entries of M_3^8 are real and the off-diagonal elements are real octonions. In the sequel I shall deal with complex Jordan algebras, where the diagonal entries are complex and, for instance, in the case of J_3^8—the complexification of M_3^8—the off-diagonal elements are complex octonions.

The complete classification of Jordan algebras on a generic field was given by Jacobson (1968), who made use of the definition of Jordan algebras in terms of the product $U_x y$, which is quadratic in x and linear in y, instead of the symmetric linear product $x \cdot y = \frac{1}{2}(xy + yx)$. The axioms for the quadratic map U_x and its linearization $V_{x,y}$, defined by $V_{x,y}z = (U_{x+z} - U_x - U_z)y$, are the following:

$$
\begin{aligned}
U_1 &= Id \\
U_x V_{y,x} &= V_{x,y} U_x \\
U_{U_x y} &= U_x U_y U_x
\end{aligned}
\tag{10}
$$

This new formulation, which is called the theory of "quadratic Jordan algebras," is equivalent to the linear approach,[6] but it reveals the essential algebraic properties of Jordan algebras:

1. If x is an invertible element of J, then U_x is invertible and $U_{x^{-1}} = U_x^{-1}$.
2. The generic norm (generalized determinant) $\mathcal{N}$ satisfies the composition $\mathcal{N}(U_x y) = \mathcal{N}(x)\mathcal{N}(y)\mathcal{N}(x)$.
3. It is possible to define inner ideals—i.e., subspaces B of J such that $U_B J \subset B$—which play the same role as the one-sided ideals in the case of an associative ring.
4. It is possible to define isotopy in terms of the quadratic map U: an isotope $J^{[v]}$ of a quadratic Jordan algebra J having an invertible element v is an algebra that has the same vector space as J but is based on the twisted product $U_x^{[v]}y = U_x U_v^{-1} y$, which shifts the unit in the isotope to $1^{[v]} = v$.

The notion of isotopy is particularly interesting, since isotopic Jordan algebras need not be isomorphic and many important properties of Jordan algebras are not affected by isotopy. The idea of including in a unique structure the concept of a Jordan algebra plus all its isotopes has led to the definition of Jordan pairs. The structure group of a Jordan algebra, as encountered in "magical supergravity," can be viewed as the group of autotopies, that is, the group of isomorphisms of J with its isotopes or, equivalently, as the automorphism group of the Jordan pair, which includes J and its isotopes.

Another generalization of Jordan algebras, based on a quadratic map, is the Jordan triple system, which can be thought of as a quadratic Jordan algebra with no unit element. I shall not write here the axioms for a Jordan triple—they are strictly analogous to those defining a Jordan pair—but I give instead a very simple example (McCrimmon, 1978). Think of the space of rectangular matrices: there is no way one can linearly multiply two such

[6]Except for fields of characteristic 2 or rings having no scalar $\frac{1}{2}$, for which the linear theory does not work (McCrimmon, 1966).

matrices x and y to get another matrix of the same type. One could do this though, through a quadratic map U such that $U_x y = xy^t x$. This is the easiest example of a Jordan triple.

A Jordan pair generalizes both the concept of a Jordan algebra and that of a Jordan triple. A Jordan pair $V = (V^+, V^-)$ (Loos, 1975) is a pair of spaces that act on each other through a quadratic map U that is such that, for $\sigma = \pm$,

$$U_{x^\sigma} y^{-\sigma} \in V^\sigma \qquad \text{for } x^\sigma \in V^\sigma; \quad y^{-\sigma} \in V^{-\sigma} \tag{11}$$

For example, we can form a Jordan pair by taking V^+ as the set of matrices $n \times m$, V^- as the set of matrices $m \times n$, and defining $U_{x^+} y^- = x^+ y^- x^+$. This generalizes the Jordan triple of the above example. Thus we have that:

Jordan triple system = Jordan pair with involution
Jordan algebra up to isotopy = Jordan pair with invertible elements.

An important feature of Jordan pairs which is relevant to this paper is the relationship between Jordan pairs and three-graded Lie algebras. A three-graded Lie algebra L is a Lie algebra that can be split into three pieces $L = L_1 + L_0 + L_{-1}$ such that $[L_i, L_j] \subset L_{i+j}$—the commutator vanishing whenever $i + j \neq 0, +1, -1$. Any three-graded Lie algebra can be obtained from a Jordan pair (McCrimmon, 1978) and, vice versa, a Jordan pair V can be obtained from L by setting

$$L_1 = V^+, \qquad L_{-1} = V^-, \qquad U_{x^+} y^- = [[x^+, y^-], x^+] \tag{12}$$

The group obtained by exponentiating L_0 is the group of automorphisms of V.

Two simple cases of three-graded Lie algebras are A_2 and C_3, the complexifications of $SU(3)$ and $Sp(3)$, respectively. In Figures 1 and 2 the root diagrams of such algebras are shown together with their split into a three-graded structure—of course, all the generators in the center of the three-graded Lie algebras, which do not appear in the root diagram, are in L_0. One can see that L_0 contains a subalgebra K ($K = A_1$, in the case of A_2; $K = A_2$, in the case of C_3). The rank (the number of generators in the center) of K is one unit lower than the rank of L. Since all the generators in the center of L are in L_0, it follows that the only generator in L_0 that is not in K is orthogonal to K—in the space of the root diagram— and hence it gives opposite charges to L_1 and L_{-1}. Therefore L_0 itself splits into the direct sum of K and the generator of an Abelian subgroup.[7]

[7]This subgroup is, for complex algebras L, a complex scale change (the subgroup, as well as its generator, is denoted in this case by $\mathbb{C}$). When taking real forms of L this subgroup becomes either $U(1)$ [or $SO(2)$] or a real scale change [or $SO(1, 1)$], according to the particular real form considered.

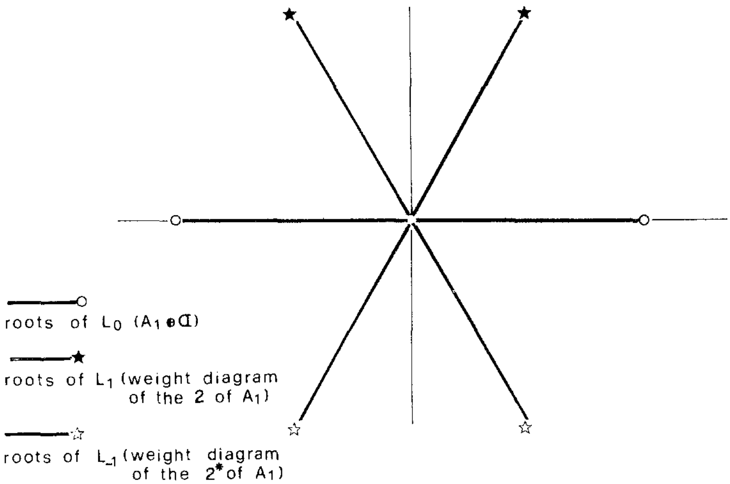

Fig. 1. Root diagram of A_2.

For instance, the subalgebra L_0 of A_2 is the direct sum of A_1, represented in the diagram by the roots of the "isospin" axis, and the "hypercharge" generator, represented in the diagram by the axis orthogonal to A_1—by abuse of notation, to be more explicit, we use for the generators in the center of A_2 the same names as those of $SU(3)$. The spaces L_1 and L_{-1} are

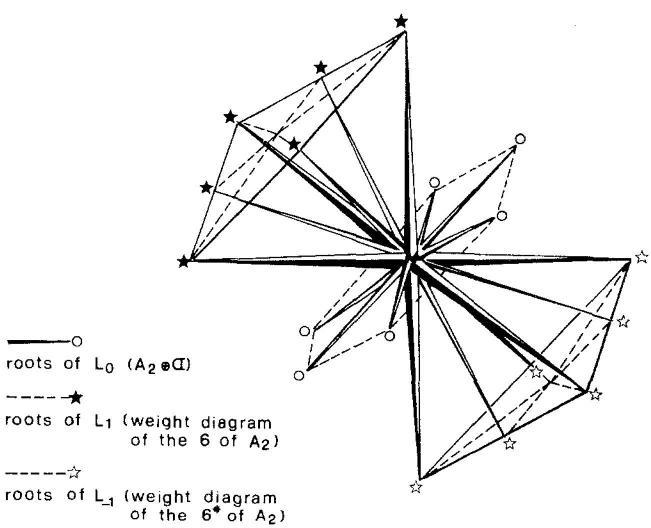

Fig. 2. Root diagram of C_3.

the carrier spaces of the representations 2 and 2* of A_1 and have opposite "hypercharge." As far as the Jordan pair content of the three-graded algebras A_2 and C_3 is concerned, it is easily shown that in the case of A_2 the Jordan pair is the doubling of a Jordan triple system (the 2 and 2* are indeed spinors, hence rectangular matrices, as in the example given above of a Jordan triple system) and in the case of C_3 it is the doubling of a Jordan algebra [in particular the Jordan algebra J_3^1 (Truini *et al.*, 1984)].

The connection between three-graded Lie algebras and Jordan pairs can be used to show the Jordan pair content of the FTMS. In fact (Truini *et al.*, 1984), referring to the "complex form" of the magic square (see Table II), we have that:

1. Any Lie algebra L in the third row is a three-graded Lie algebra (we have just shown the case of C_3 (third row, first column) as an example).
2. The subalgebra L_0 of L is the direct sum of the Lie algebra appearing in the second row of the FTMS (at the same column of L) with the generator of a complex scale change.
3. The spaces L_1 and L_{-1} (namely the Jordan pair) are two copies of the complex Jordan algebra shown on top of the FTMS at the same column of L and they are the carrier spaces of two representations, one conjugate to the other, of the group generated by L_0.

Since L_0 generates the group of automorphisms of the Jordan pair, the second row of the magic square is to be viewed as formed by the Lie algebras generating the automorphism groups of the Jordan pairs (L_1, L_{-1}). The three-grading of the Lie algebras in the third row of the FTMS is shown in Table III (Truini *et al.*, 1984).

To make contact with "magical supergravity," we must consider a real form of the FTMS, in particular the real form shown in Table I. In this real case also the algebras in the third row are three-graded Lie algebras, but the Jordan pair that we had in the complex case is restricted (since the number of parameters is halved) into one of the following cases, depending upon the real form considered [Truini *et al.* (1984); see also Jacobson (1971) for the exceptional case]:

1. A Jordan pair with involution (which is equivalent to a Jordan triple system.
2. A pairing of a real Jordan algebra (which is equivalent to a real Jordan algebra up to isotopy).

The parameter counting shows that the above alternative depends upon the fact that the restriction to the real form of L_1 (and L_{-1}) be a complex

Table III. Three-Grading for the Third Row of the FTMS[a]

Algebras	*Complex dimension*
$C_3 = J_3^1 \oplus (A_2 \oplus \mathbb{C}) \oplus J_3^{1*}$	$21 = 6 + (8+1) + 6^*$
$A_5 = J_3^2 \oplus (A_2 \oplus A_2 \oplus \mathbb{C}) \oplus J_3^{2*}$	$35 = (3\times 3) + (8+8+1) + (3\times 3)^*$
$D_6 = J_3^4 \oplus (A_5 \oplus \mathbb{C}) \oplus J_3^{4*}$	$66 = 15 + (35+1) + 15^*$
$E_7 = J_3^8 \oplus (E_6 \oplus \mathbb{C}) \oplus J_3^{8*}$	$133 = 27 + (78+1) + 27^*$

[a]The asterisk denotes the conjugate representation.

or a real carrier space for the representation of the group generated by the real form of L_0.

To illustrate what was just said, consider again the example of A_2. Although A_2 is not an algebra in the third row of the FMTS, it is very simple to work with and shows just as well the essence of what I am asserting. What we have learned is that the three-grading of A_2 is the direct sum of $L_0 = A_1 \oplus \mathbb{C}$ (by $\mathbb{C}$ I indicate here the generator of a complex scale change), $L_1 = 2$ (the two-dimensional representation of the group generated by A_1), and $L_{-1} = 2^*$. Let us consider now the compact real form of A_2: $SU(3)$. The subalgebra L_0 becomes $SU(2) \oplus U(1)$. Since the two-dimensional representation of $SU(2)$ has a complex carrier space, it is clear, by counting the parameters, that an element in the Jordan pair must be composed of an element in the 2 of $SU(2)$ and its conjugate (not an independent one) in the 2^*. In fact, writing a generic element of the algebra of $SU(3)$[8] as the skew-symmetric matrix

$$\begin{pmatrix} \alpha_1 & a & b \\ -a^* & \alpha_2 & c \\ -b^* & -c^* & \alpha_3 \end{pmatrix} \tag{13}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}, \qquad a, b, c \in \mathbb{C}$$

one can take a generic element of L_0

$$\begin{pmatrix} \alpha_1 & a & 0 \\ -a^* & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{14}$$

and the 2 and 2^* are

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b^* & -c^* & 0 \end{pmatrix} \tag{15}$$

[8]I abuse the notation for groups and corresponding algebras by denoting them in the same way: it should be clear by the context which one I refer to.

We see from (13)–(15) that the elements of the 2 and the 2^* are not separately elements of $SU(3)$, but the pair $(x, -x^*)$, where x is an element of the 2 of $SU(2)$, does belong to $SU(3)$.

If we had taken $Sl(3, \mathbb{R})$ as a real form of A_2, then a generic element of L would be a generic, real, traceless matrix and the algebra L_0 would be $Gl(2, \mathbb{R})$. The conjugation between the 2 and the 2^* of $Gl(2, \mathbb{R})$ is simply the transposition and an element x in the 2 or y^* in the 2^* as well as a pair (x, y^*), with y independent of x, all belong to $Sl(3, \mathbb{R})$.

This trivial example shows how naturally the Jordan pair language enters the theory of three-graded Lie algebras. In the sequel I show how Jordan pairs describe the scalar manifold in $d = 4$ "magical supergravity," where the global symmetry group is generated by three-graded Lie algebras.

5. JORDAN PAIRS IN SUPERGRAVITY

Let me go back to the reduction of "magical supergravity" to four dimensions. The scalars parametrize the following coset spaces G/H:

$$\begin{aligned} &Sp(3, \mathbb{R})/SU(3) \times U(1), \qquad SU(3,3)/SU(3) \times SU(3) \times U(1) \\ &SO^*(12)/SU(6) \times U(1), \qquad E_{7(-25)}/E_6 \times U(1) \end{aligned} \tag{16}$$

Since G is generated by a three-graded Lie algebra L (third row of the FTMS) and the subalgebra L_0 in the three grading of L generates H (Truini *et al.*, 1984), it follows that L/L_0 is a Jordan pair, which means that the tangent spaces to the manifolds (16) are Jordan pairs. Moreover, because L_1 and L_{-1} are complex representations of the group H, it follows from the previous section that the Jordan pair (L_1, L_{-1}) is formed by elements linked by an involution, which effectively makes the Jordan pair a Jordan triple system. It follows from the fact that the group H is the maximal compact subgroup of G that the involution exchanging L_1 with L_{-1} is the complex conjugation (Truini *et al.*, 1984). The Jordan pair is therefore made up of elements (x, x^*), where $x \in J_3^i$, $i = 1, 2, 4, 8$ (the complexification of M_3^i), and the asterisk denotes the conjugation in the complex field which complexifies the real Jordan algebra M_3^i. The compact group H is the automorphism group of such pairs.

Among the "magical" cases, those associated with the exceptional groups are the most interesting ones. They are, in fact, the only ones that cannot be obtained as truncation of $N = 8$ supergravity theories (Günaydin *et al.*, 1984). In this sense they are a maximal extension of an $N = 2$ theory. In this context, GST present in their papers another exceptional case in four dimensions in which the manifold parametrized by the scalars is the coset

$$E_{6(-14)}/SO(10) \times SO(2) \tag{17}$$

I will not go into a detailed discussion of this case from the point of view of the Jordan pairs, for which I refer to Truini *et al.* (1984). I just report here the result that the points in the geometry (17) can be regarded as idempotent elements of a Jordan pair.[9]

In a previous paper (Truini and Biedenharn, 1982) we examined the geometry of

$$E_{6,0}/SO(10)\times SO(2) \tag{18}$$

where $E_{6,0}$ is the compact form of E_6. This geometry has been viewed as a planar geometry (very close to a projective plane) whose points have been classified as a specific class of idempotents of the Jordan pair embedded in the three grading of E_7, namely the Jordan pair (J_3^8, J_3^{8*}). In a sense we have found for Jordan pairs an analogous situation to that of real Jordan algebras of 3×3 Hermitian matrices, for which the set of trace-one idempotents (the projectors associated with pure states in quantum mechanics) form a projective plane.

The case of the coset (17) is quite different from the case (18), due to the fact that $E_{6(-14)}$, being noncompact, has different strata. This implies that $E_{6(-14)}$ cannot act transitively on the same class of idempotents as the one that forms the geometry (18). An orbit of $E_{6(-14)}$ in the stratum having $SO(10)\times SO(2)$ as stability group is like a two-sheeted hyperboloid and the description of the coset (17) by use of idempotents is the analogue of parametrizing a hyperboloid by means of projective (inhomogeneous) coordinates. It is well known that this does not turn the coset into a projective plane. Analogously, the geometries (18), which is as close to a projective plane as can be (Truini, and Biedenharn, 1982), and (17) are essentially different, but they both can be described using idempotents of the Jordan pair $(J_3^8, \tilde{J}_3^8)$, the tilde denoting the suitable conjugation for the real form of E_6.

The scalar fields that parametrize the coset (17) can thus be viewed as elements of a Jordan pair. An advantage of this formulation is that the action of the group $E_{6(-14)}$ on the points of the coset can be easily obtained by the action of the three-grading on the Jordan pair, and this in turn is the action of E_7 on its own generators.

The coset (17) can be viewed as related to the fourth column of the magic square, since its points are idempotent elements of a Jordan pair related to real forms of E_6 and E_7, which are Lie algebras of the fourth column of the FTMS. One can extend, therefore, the construction of cosets out of idempotents of a Jordan pair to the Jordan pairs related to the first three columns of the magic square. In this way one would get the cosets

[9]An idempotent of a Jordan pair is a pair (x, y) such that $U_x y = x$, $U_y x = y$.

(Truini *et al.*, 1984)

$$SU(2,1)/SU(2)\times U(1)$$
$$SU(2,1)\times SU(2,1)/SU(2)\times SU(2)\times U(1) \qquad (19)$$
$$SU(4,2)/SU(4)\times SU(2)\times U(1)$$

Hence these coset spaces also can be considered as possible scalar manifolds for a nonexceptional $N=2$, $d=4$ supergravity theory. The manifolds (19) can all be embedded into the manifold (17) and are obviously easier to work with. One could therefore start examining these cosets as an approach to the study of the manifold (17), which already reveals essential features of this more complicated structure.

Remark. I emphasize that there is a substantial difference in regarding the geometries (17) and (19), on the one side, and (6), on the other side, as Jordan pairs. In the former case the (idempotent) elements of the Jordan pair parametrize the coset; in the latter case the elements of the Jordan pair sit in the tangent space of the coset, since in this latter case the Jordan pair is the modulo of two Lie algebras and not of the corresponding Lie groups.

6. CONCLUSIONS

A belief that many theoretical physicists share is that a good theory has to be characterized by mathematical beauty and simplicity. I do have the feeling that this should be so, not quite because what theoretical physics tries to describe is beautiful and simple, but rather because it is a duty of the theoretical physicist to find a language to describe the theory that makes it look beautiful and simple. The importance of finding such a language lies in the fact that, if a language has such a power of synthesis, it can easily unveil deeper structures in the theory, thus opening the way for further progress.

From this point of view the language of Jordan pairs seems to have several important features, which I hope to have sufficiently emphasized in the present paper:

1. It is a most natural description, in the framework of the Jordan theory, for a system that includes both Jordan triples and Jordan algebras.
2. It very nicely preserves the Lie albegra structure whenever a link is made between Jordan pairs and three-graded Lie algebras.
3. It unifies the description of two different cases of $d=4$ magical supergravity [see (6) and (17), (19)].

Fubini, S., and Nicolai, H. (1985). *Physics Letters*, **155B**, 369.
Green, M. B., and Schwarz, J. H. (1984). *Physics Letters*, **149B**, 117.
Günaydin, M., Piron, C., and Ruegg, M. (1978). *Communications in Mathematical Physics*, **61**, 69.
Günaydin, M., Sierra, G., and Townsend, P. K. (1983). *Physics Letters*, **133B**, 72.
Günaydin, M., Sierra, G., and Townsend, P. K. (1984). *Nuclear Physics B*, **242**, 244.
Gürsey, F. (1975). In *International Symposium of Mathematical Problems in Theoretical Physics*, A. Araki, ed. Springer, New York.
Gürsey, F. (1977). In *Group Theoretical Methods in Physics*, R. T. Sharp and B. Kolman, eds., Academic Press, New York.
Gürsey, F. (1978). In *Second Johns Hopkins Workshop on Current Problems in High Energy Particle Theory*, G. Damokos and S. Kovesi-Damokos, eds., Johns Hopkins University, Baltimore, Maryland.
Jacobson, N. (1968). *Structure and Representation of Jordan Algebras*, American Mathematical Society, Providence, Rhode Island.
Jacobson, N. (1971). In *Lecture Notes in Pure and Applied Mathematics*, p. 1, Dekker, New York.
Jordan, P. (1932). *Nachrichten Gesellschaft der Wissenschaften, Gottingen*, **1932**, 569.
Jordan, P., von Neumann, J., and Wigner, E. P. (1934). *Annals of Mathematics*, **35**, 29.
Loos, O. (1975). In *Lecture Notes in Mathematics*, p. 460, Springer, Berlin.
McCrimmon, K. (1966). *Proceedings of the National Academy of Sciences of the United States of America*, **56**, 1072.
McCrimmon, K. (1978). *Bulletin of the American Mathematical Society*, **84**(4), 612.
Schafer, R. D. (1966). *An Introduction to Nonassociative Algebras*, Academic Press, New York.
Tits, J. (1955). *Académie Royale de Belgique. Classe des Sciences. Mémoires* (3).
Truini, P., and Biedenharn, L. C. (1982). *Journal of Mathematical Physics*, **23**(7), 1327.
Truini, P., Olivieri, G., and Biedenharm, L. C. (1984). Genoa preprint 84/12.
Truini, P., Olivieri, G., and Biedenharn, L. C. (1985). *Letters in Mathematical Physics*, **9**, 255.
Van Nieuwenhuizen, P. (1981). *Physics Reports*, **68C**, 191.